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On the Solution of a Class of Inverse Evolution Problems by a Bellman-Adomian Method

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Abstract. A class of inverse initial boundary value problems with one of the two boundary conditions unknown and given solution in some interior point can be solved by a joint application of Bellman's quadrature method to reduce the partial differential equation to a system of ordinary differential equation and of Adomian's decomposition method to solve analitically such a system of equations.

1. Introduction

The practical solution of inverse problems is of relevant interest in applied sciences [1,2]. The Author has shown in a recent paper [3] how several inverse initial-boundary value problems in one space dimension can be solved by a suitable joint application of Bellman's differential quadrature method [4] and Adomian's decomposition method [5].

The first method is used to space interpolate the solution transforming the original i.b.v. problem for p.d. equations into the solution of an i.v. problem for a system of o.d. equations. The second method is used to obtain simple analytic (approximated) solutions of the system of o.d. equations. Obtaining analytical solutions allows, as we shall see, the solution of the class of inverse problems described in the next section.

The application (joint) of the two methods has been recently used to solve some nonlinear direct problems both in the deterministic [6] and the stochastic case [7].

This paper is organized in three sections. After this introduction the next section deals with the description of the class of inverse problems which is here dealt with and provides the detailled description of the mathematical method. The last section deals with a simple application suitable to show the practical application of the method itself.

2. The Mathematical Method.

Consider the following mathematical evolution problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) \tag{1}$$

where $\quad u = u(t,x) : [0,T] \times [0,1] \longrightarrow \mathrm{R} \quad$ is joined to the conditions

$$\begin{aligned} u_0 &= u_0(x) = u(x,t), && \forall x \in [0,1], \\ u_{b0} &= u_{b0}(t) = u(0,t), && \forall t \in [0,T] \\ u_k &= u_k(t) = u(x_k,t), && 0 < x_k < 1, \forall t \in [0,T] \end{aligned} \tag{2}$$

where $u_k = \sum_{h=0}^{m} c_h t^h$.

Such a problem corresponds to an inverse evolution problem such that the boundary condition at $x = 1$ is unknown, but the time evolution of u at $x = x_k$ is given by some direct measure.

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The mathematical problem consists in finding the time evolution of the dependent variable $u = u(t, x)$ and in particular of u at $x = 1$. This target will be sought for by means of a suitable development of the so called Bellman-Adomian method. The method, in particular, developes some idea proposed in [3] and can be organized according to the following steps:

1. Discretize the space variable by n nodal points $x_1 = 0, x_2, ..., x_k, ..., x_n = 1$ and interpolate the solution of problem (1-2) by a Lagrangian-type interpolation

$$u(t,x) \simeq \sum_{i=1}^{n} p_i(x) u_i(t; x = x_i) \tag{3}$$

where the p_i are lagrange polynomials also reported in [7].

2. Express the time evolution of the variable u at the boundary $x = 1$ by a time-power expansion

$$u_n = \sum_{h=0}^{m} \gamma_h t^k = \sum_{h=0}^{m} \lambda^h \gamma_k t^h, \qquad \lambda = 1 \tag{4}$$

Compute the space derivative $\partial^2 u/\partial x^2$ through eq. (3)

$$\frac{\partial^2 u}{\partial x^2}(t, x = x_i) = \sum_{j=1}^{n} b_{ij} u_j(t) \tag{5}$$

where $b_{ij}{=}dp_j/dx$ $(x = x_i)$, and substitute (4-5) into eq. (1) obtaining a system of (n-2) ordinary differential equations wich can be written in integral form as

$$u_i(t) = u_{i0} + b_{in} \sum_{h=0}^{m} \frac{\lambda^h \gamma_h}{h+1} t^{h+1} + \lambda \int_0^t \left(\sum_{j=2}^{n-1} b_{ij} u_j(s) - f(u_i(s)) \right) ds. \tag{6}$$

for $i = 2, ..., n-1$ and $u_1 = u_{b0}$.

3. Solve eq. (6) by Adomian's decomposition method [3] which, as discussed in Chap. 3 of ref [8], provides the solution in the form

$$u_i(t) \simeq \sum_{h=0}^{m} \beta_{ih} t^h, \quad \beta_{ih} = \beta_{ih}(\gamma_0, ..., \gamma_{h-1}) \tag{7}$$

4. Equate the terms with the same power of t referred to the nodal point $x_k, c_h = \beta_{kh}$, to provide, recalling that the terms β_{kh} is a function of the unknowns γ_h, the actual expression of the γ_h so that also the β_{ih} are characterized. The solution of problem (1-3) can then be written as

$$u(t,x) \simeq \sum_{i=1}^{n} \sum_{h=0}^{m} p_i(x) \beta_{ih} t^h \tag{8}$$

Remark 1: The application (technical) of the decomposition method has been described several times elsewhere [9] so that it is not reported here.

Remark 2: The method appears to be very practical and efficient for short time intervals. For large time intervals accuracy is obtained as in [7], discretizing the time interval and using the solution obtained by decomposition method as an algorithm for time integration which provides, as discussed in Chapter 3 of ref [8], a solution continuous in time.

Remark 3: It is a technical problem to show that the method can be applied to more general cases than the ones indicated in [1]. In fact the differential quadrature method is very flexible (in one space dimension) and can be applied in the case in very general class of equatios and for very general boundary conditions.

3. Application and Discussion

Consider now the following particular expression of eq. (1)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha u^2; \qquad u_0 = \epsilon x, \quad u_{b0} = 0, \quad u_k = \epsilon x_k - \sum_{h=1}^{m} c_h t^h \tag{9}$$

where c_h are given constants. After the discretization interpolation described in the preceding section, the following system of ordinary differential equations is obtained

$$\lambda = 1: \quad u_i(t) = u_{i0} + \lambda \int_0^t \left(\sum_{j=2}^{n-1} b_{ij} u_j + \sum_{h=0}^{m} b_{in} \gamma_h t^h \lambda^h - \alpha u_i^2 \right) ds \tag{10}$$

the solution of this equation can be sought for in the form given by eq. (7) and the practical application of the decomposition method yields the following expressions of the first terms of the decomposition

$$u_i = u_i^{(0)} + u_i^{(1)} + u_i^{(2)} + \ldots \tag{11}$$

$$u_i^{(0)} = u_{i0} \tag{12a}$$

$$u_i^{(1)} = \left(\sum_{j=2}^{n-1} b_{ij} u_{j0} + b_{in} \gamma_0 - \alpha u_{i0}^2 \right) t = \beta_{i1} t \tag{12b}$$

$$u_i^{(2)} = \left(\sum_{j=2}^{h-1} b_{ij} \beta_{j1} + b_{in} \gamma_1 - 2\alpha u_{i0} \beta_{i1} \right) \frac{t^2}{2} = \beta_{12} t^2 \tag{12c}$$

and so on. Hence

$$\gamma_0 = \frac{1}{b_{kn}} \left[c_1 - \sum_{j=2}^{n-1} b_{kj} u_{j0} + \alpha u_{k0}^2 \right] \tag{13a}$$

$$\gamma_1 = \frac{1}{b_k n} \left[2c_2 - \sum_{j=2}^{n} b_{kj} \beta_{j1} + 2\alpha u_{k0} \beta_{k1} \right] \tag{13b}$$

and so on.

Note that eq. (13a) defines γ_0 which can be cast into (12b) so that β_{i1} is defined. Then eq. (13a) defines γ_1 which can be cast into (12c) in order to define β_{i2} and so on.

Applying the method is a routine and generally provides accurate results using a few terms of the decomposition. The content of Remark 2 has to be carefully considered for practical applications. However several tests show the efficiency of the method.

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